This formula gives a result somewhat greater than 90 radii of the earth. Assuming 90 radii as correct, the distance the spirits fell in the first second is,

$$x : 16.1 :: 1^2 : 90^2$$

x being .0237 inch.

Assuming that 460,000 radii was correct, the distance fallen in the first second would be .000,000,000,007,2 inch.

NOTE ON PROBLEM 95.

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The problem arose in a discussion carried on in the *Nation*, Vol. 68, page 376, between Mr. C. S. Peirce and myself, relating to the validity of an argument given by Galileo and intended to refute the hypothesis that the velocity of a falling body varies as the distance described from a state of rest. Galileo says: "If the velocity with which a body overcomes four yards is double the velocity with which it passed over the first two yards, then the times necessary for these processes must be equal; but four yards can be overcome in the same time as two yards only if there is an instantaneous motion." Mr. Peirce argues that Galileo's reasoning is sound, a claim which I cannot admit.

The assumption that the velocity shall be proportional to the distance described from the state of rest can be expressed by the formula

$$\frac{ds}{dt}=as, \text{ where } a \text{ is a constant.}$$

Hence the acceleration is $\dfrac{d^2s}{dt^2}=a^2s$. Now initially the distance passed over is zero, *i. e.* $s=0$. Hence the initial accelleration is zero and the body can never begin to move. This conclusion stands even when a is infinitely large, for when absolute zero is a factor, then the product must be zero, no matter how large the other factor may be.* This is the point on which the whole discussion turns. Since Galileo concludes that instantaneous motion is the result, when really there can be no motion at all, his reasoning is fallacious.

But Peirce argues that Galileo used both assumptions stated in the problem, viz., (1) $\dfrac{ds}{dt}=as$, and (2) t finite for a finite distance. Peirce says: ". . . . the solution of the differential equation $\dfrac{ds}{dt}=as$ is $s=Ce^{at}$. In order that s and t should both be zero together, C must be infinitesimal. Then, for a finite value of s, either a or t must be infinite. That is, either the acquired velocity or the time of fall must be infinite. Galileo's argument first adduces the fact that the time is finite, and on that assumption concludes that the hypothesis would in-

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*"In putting together *naughts* to arrive at 1, *we never make any way at all;* the second thousand processes gives no more than the first." DE MORGAN.

volve an infinite acquired velocity, which is absurd." In this way Peirce justifies Galileo's conclusion.

The error in Peirce's reasoning seems to me perfectly apparent. When $s=0$ he takes C to be an infinitesimal, while really C is an absolute zero. When s and t are both zero, e^{at} is not zero, hence e^{at}, multiplied by an infinitesimal, cannot be equal to absolute zero. An infinitesimal is a variable whose limit is zero, but the variable never reaches its limit. If we consider an infinitesimal an extremely small quantity, we must still remember that it is a quantity. Now, if C is absolute zero, then s can never be different from zero, no matter how large e^{at} may be.

It is easy to illustrate my conclusion by physical examples. A particle is placed in a smooth tube which revolves horizontally about an axis through its center. With what velocity will the particle move? The only force impelling the particle along the tube is the centrifugal force due to rotation. Hence we have $\frac{d^2s}{dt^2}=w^2s$ and $\frac{ds}{dt}=ws$, where w is the uniform angular velocity. Here the velocity is proportional to the distance from the axis. Suppose now that the particle lies initially at rest *in the axis*. Will it begin to move? There is no reason why it should move one way any more than the other.

The two assumptions in our problem are contraries. The first excludes the possibility of motion; the second declares that motion exists. From assumptions that are contraries no conclusion can be drawn.

DIOPHANTINE ANALYSIS.

80. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Find three square numbers whose reciprocals form an arithmetical progression.

I. **Solution by the PROPOSER.**

When three numbers are in arithmetical progression, the products of these numbers, taken two at a time, will give three numbers whose *reciprocals* are in arithmetical progression.

Let $x-a$, x, and $x+a$=three numbers in arithmetical progression. Then $x(x-a)$, $(x-a)(x+a)$, and $x(x+a)$=the three numbers whose reciprocals are in arithmetical progression.

$$\text{For, } \frac{1}{x(x-a)}-\frac{1}{(x-a)(x+a)}=\frac{1}{(x-a)(x+a)}-\frac{1}{x(x+a)}=\frac{a}{x(x-a)(x+a)}.$$

The general values for three *squares* in arithmetical progression are $(p^2-q^2-2pq)^2$, $(p^2+q^2)^2$, and $(p^2-q^2+2pq)^2$, where $x=(p^2+q^2)^2$, and a the common difference$=4pq(p^2-q^2)$.

Put $p=2$ and $q=1$; then 1^2, 5^2, and 7^2 are three squares in arithmetical progression. Whence the three squares whose reciprocals form an arithmetical